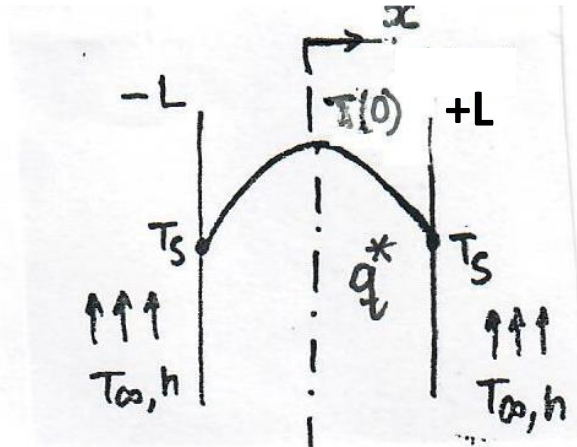


Heat and Mass transfer

1-D Steady State Conduction in Various Geometry

Plane Wall with Uniformly Distributed Heat source



$$\frac{d^2 T}{dx^2} + \frac{q^*}{k} = 0 \quad \text{or} \quad \frac{dT}{dx} = -\frac{q^*}{k}x + c_1$$

Or

$$T = c_1 x - \frac{q^* x^2}{2k} + c_2 \quad (1)$$

At $x = -L$, $T = T_{s1}$ and at $x = +L$, $T = T_{s2}$

$$c_1 = \frac{T_{s2} - T_{s1}}{2L} \quad \text{and} \quad c_2 = \frac{q^* L^2}{2k} + \frac{T_{s1} + T_{s2}}{2}$$

$$T(x) = \frac{q^* L^2}{2k} \left[1 - \frac{x^2}{L^2} \right] + \frac{T_{s2} - T_{s1}}{2} \frac{x}{L} + \frac{T_{s1} + T_{s2}}{2} \quad (2)$$

If $T_{s1} = T_{s2} = T_s$, then

$$T(x) = \frac{q^* L^2}{2k} \left[1 - \frac{x^2}{L^2} \right] + T_s \quad (3)$$

The maximum temperature is at $x = 0$, given by

$$T(0) = \frac{q^* L^2}{2k} + T_s \quad (4)$$

A common situation is one for which, it is the temperature of the adjoining fluid is known, $= T_\infty$. In such a case

$$-k \frac{dT}{dx} \Big|_{x=L} = h(T_s - T_\infty) \quad (5)$$

From equations (3) and (5),

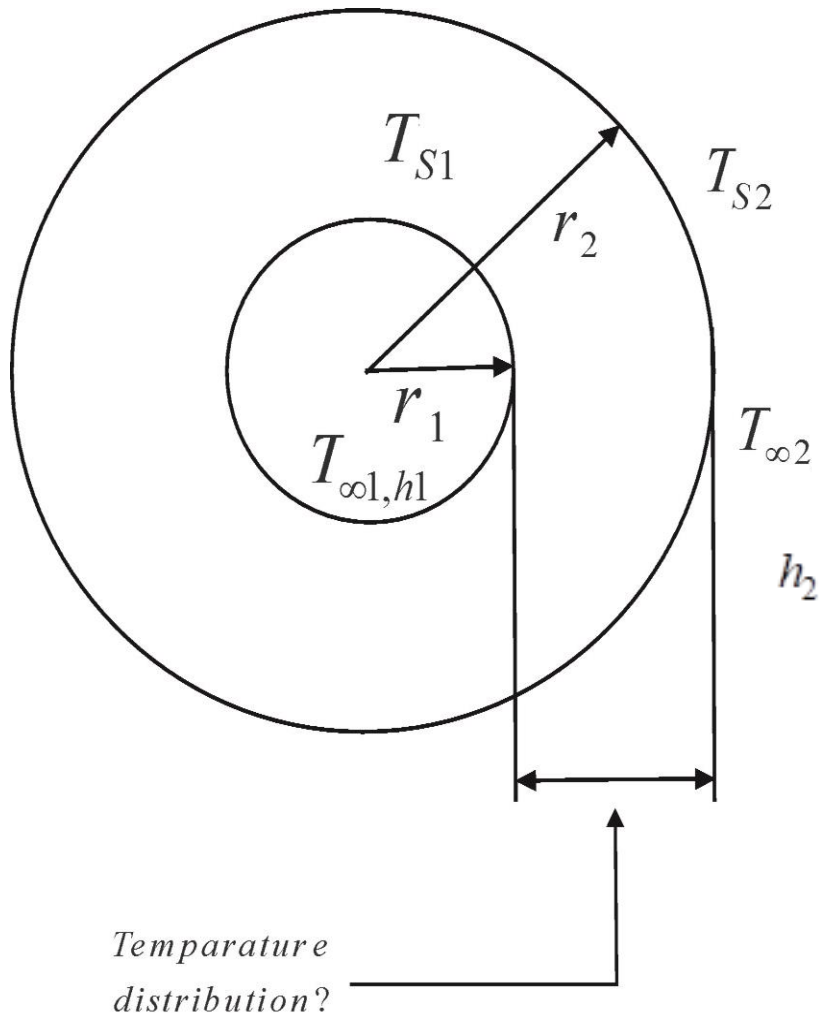
$$T_s = T_\infty + \frac{q^* L}{h} \quad (6)$$

Find out,

$$\frac{T(x) - T(0)}{T_s - T(0)}$$

Analysis in Radial Systems

Hollow Cylinder



Consider a hollow cylinder whose inner and outer surfaces are exposed to different temperatures.

For 1-D steady state without heat generation, the governing equation is

$$\frac{1}{r} \cdot \frac{d}{dr} \left(k r \frac{dT}{dr} \right) = 0 \quad (1)$$

The rate at which heat is conducted across any cylindrical surface in the solid

$$\begin{aligned} q_r &= -k (2\pi r L) \frac{dT}{dr} \\ &= -2\pi L \left(k r \frac{dT}{dr} \right) \end{aligned} \quad (2)$$

From the above two equations it follows that conduction heat transfer rate q_r is constant in radial direction

From (1) we get:

$$kr \frac{dT}{dr} = c$$

$$\text{or, } \frac{dT}{dr} = \frac{c_1}{r}$$

$$\text{or, } T = c_1 \ln r + c_2$$

To obtain the constants of integration, c_1 and c_2 we introduce the boundary conditions at $r = r_1$ and $r = r_2$

$$r = r_1, \quad T = T_{S1}$$

$$\underline{r = r_2, \quad T = T_{S2}}$$

For the Hollow Cylinder

$$T_{S1} = c_1 \ln r_1 + c_2$$

$$\underline{T_{S2} = c_1 \ln r_2 + c_2}$$

$$T_{S1} - T_{S2} = c_1 \ln \frac{r_1}{r_2} \rightarrow c_1 = \frac{T_{S1} - T_{S2}}{\ln \frac{r_1}{r_2}}$$

$$c_2 = T_{S2} - c_1 \ln r_2 = T_{S2} - \frac{(T_{S1} - T_{S2})}{\ln \frac{r_1}{r_2}} \ln r_2$$

$$T = \frac{T_{S1} - T_{S2}}{\ln \left(\frac{r_1}{r_2} \right)} \ln r - \frac{(T_{S1} - T_{S2})}{\ln \left(\frac{r_1}{r_2} \right)} \ln r_2 + T_{S2}$$

$$\text{or, } T(r) = \frac{(T_{S1} - T_{S2})}{\ln \left(\frac{r_1}{r_2} \right)} \ln \left(\frac{r}{r_2} \right) + T_{S2}$$

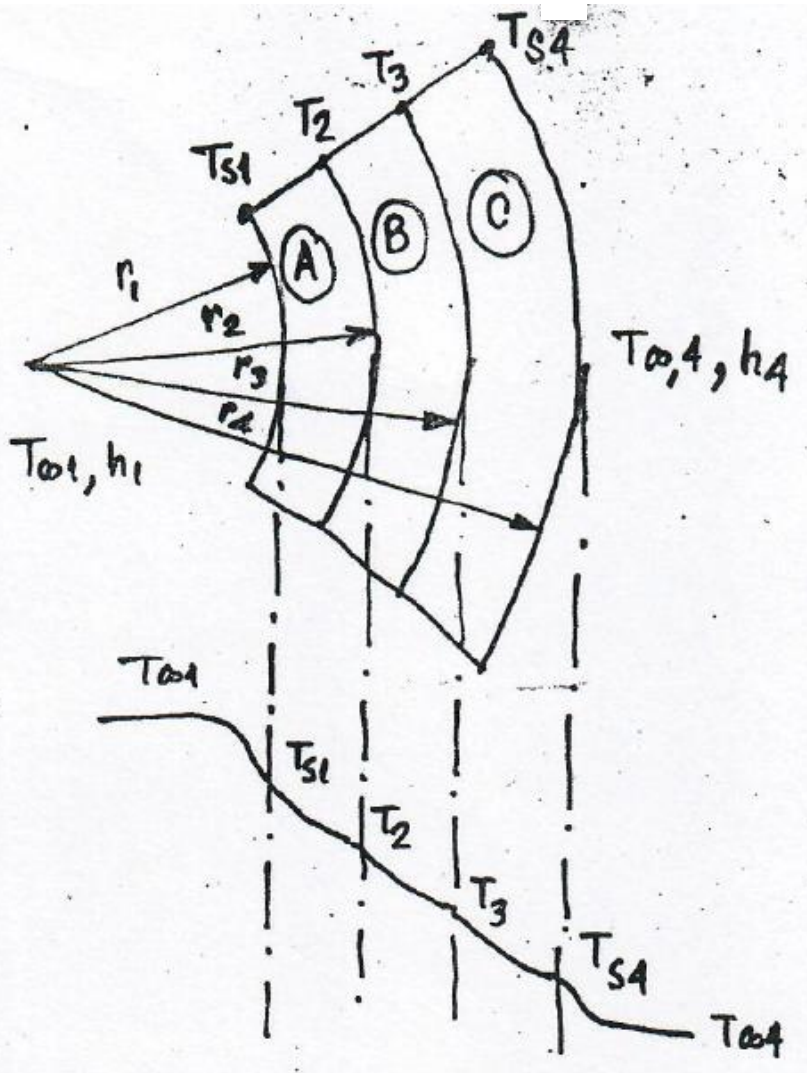
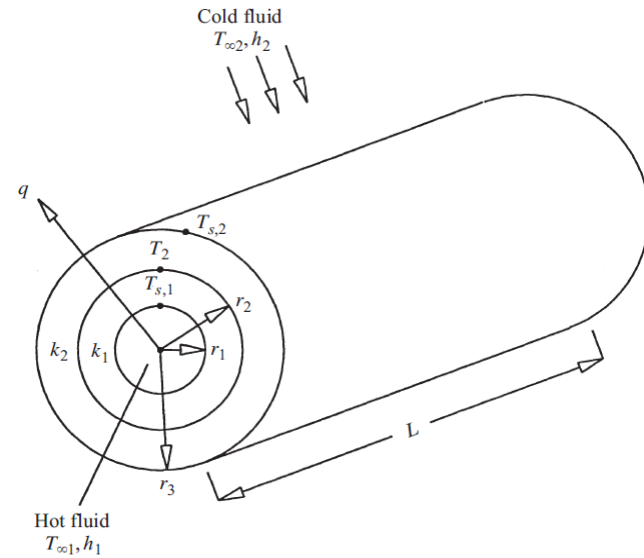
The temperature distribution associated with radial conduction through a cylinder is logarithmic, - not linear, as it for the plane wall under the same conditions. Now, the heat transfer rate:

$$q_r = -kA \frac{dT}{dr} = -k(2\pi rL) \frac{(T_{S1} - T_{S2})}{\ln\left(\frac{r_1}{r_2}\right)} \cdot \frac{r_2}{r} \cdot \frac{1}{r_2}$$

$$\text{or, } q_r = -2\pi kL \frac{(T_{S1} - T_{S2})}{\ln\left(\frac{r_1}{r_2}\right)} = \frac{2\pi kL(T_{S1} - T_{S2})}{\ln\left(\frac{r_2}{r_1}\right)}$$

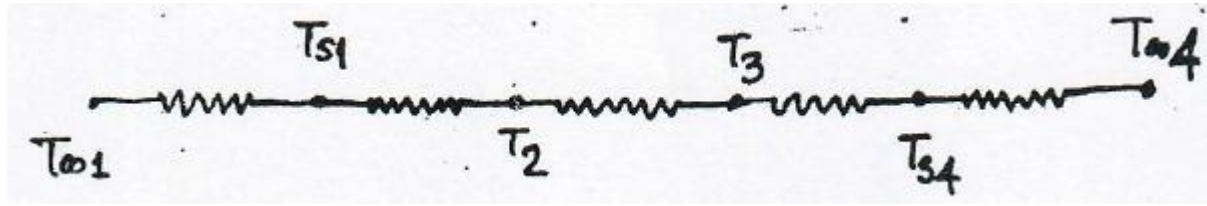
From this result it is evident that, for conduction in hollow cylinders, the thermal resistance is of the form

$$R_{t,cond} = \frac{T_{S1} - T_{S2}}{q_r} = \frac{\ln(r_2/r_1)}{2\pi kL}$$



Consider now the composite system shown in the figure. The heat transfer rate may be expressed as

$$q_r = \frac{T_{\infty 1} - T_{\infty 4}}{\left\{ \frac{1}{2\pi r_1 L h_1} + \frac{\ln(r_2/r_1)}{2\pi k_A L} + \frac{\ln(r_3/r_2)}{2\pi k_B L} + \frac{\ln(r_4/r_3)}{2\pi k_C L} + \frac{1}{2\pi r_4 L h_4} \right\}}$$



The forgoing result may be expressed in terms of an Overall heat transfer coefficient. That is

$$q_r = \frac{T_{\infty 1} - T_{\infty 4}}{R_{tot}} = U_1 A_1 (T_{\infty 1} - T_{\infty 4})$$

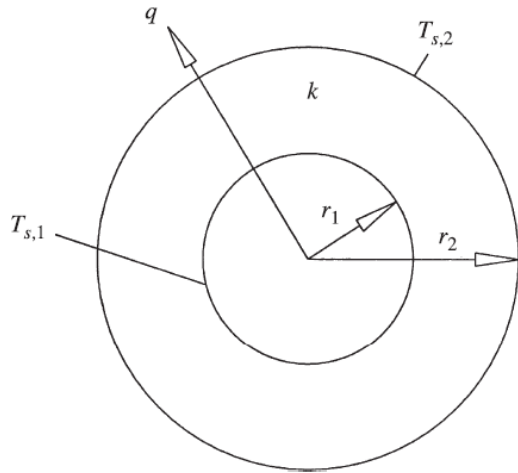
Where $A_1 = 2\pi r_1 L$; now

$$U_1 = \frac{1}{\frac{1}{h_1} + \frac{r_1}{k_A} \ln\left(\frac{r_2}{r_1}\right) + \frac{r_1}{k_B} \ln\left(\frac{r_3}{r_2}\right) + \frac{r_1}{k_C} \ln\left(\frac{r_4}{r_3}\right) + \frac{r_1}{r_4} \cdot \frac{1}{h_4}}$$

This is defined on the basis of inside surface area A_1 of the composite cylinder. The overall heat transfer coefficient may also be defined in terms of A_4 or any of the intermediate areas. And we can write

$$U_1 A_1 = U_2 A_2 = U_3 A_3 = U_4 A_4 = \left(\sum R_t\right)^{-1}$$

Hollow Sphere



$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(k r^2 \frac{dT}{dr} \right) = 0 \quad (1)$$

According to Fourier's law $q_r = -kA \frac{dT}{dr}$

$$q_r = -k4\pi r^2 \frac{dT}{dr} \quad (2)$$

Combination of (1) and (2) dictates $q_r = \text{const}$

From (1) $kr^2 \frac{dT}{dr} = c \quad \text{or} \quad \frac{dT}{dr} = \frac{c_1}{r^2}$

$$\text{or, } T(r) = -\frac{c_1}{r} + c_2$$

$$\left. \begin{array}{l} \text{at } r = r_1, \quad T = T_{S1} : T_{S1} = -c_1/r_1 + c_2 \\ \text{at } r = r_2, \quad T = T_{S2} : T_{S2} = -c_1/r_2 + c_2 \end{array} \right\} T_{S1} - T_{S2} = c_1 \left(\frac{1}{r_2} - \frac{1}{r_1} \right)$$

$$\text{or } c_1 = \frac{T_{S1} - T_{S2}}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)} \left. \vphantom{\frac{T_{S1} - T_{S2}}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}} \right\} \quad c_2 = T_{S1} + \frac{T_{S1} - T_{S2}}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)} \cdot \frac{1}{r_1}$$

$$T(r) = -\frac{T_{S1} - T_{S2}}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)} \cdot \frac{1}{r} + T_{S1} + \frac{T_{S1} - T_{S2}}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)} \cdot \frac{1}{r_1}$$

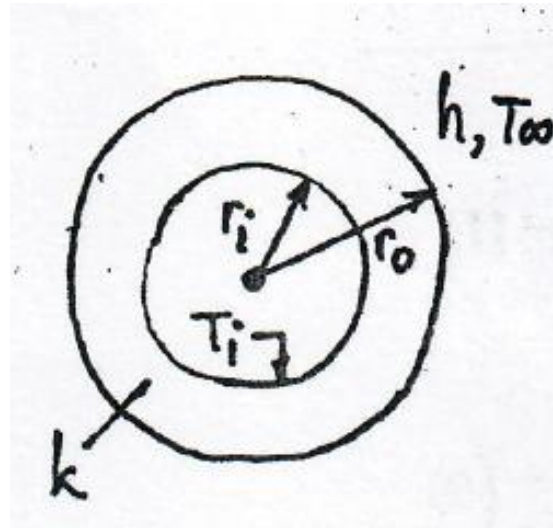
$$T(r) = T_{S1} - \frac{(T_{S1} - T_{S2})}{\left(1 - \frac{r_1}{r_2}\right)} + \frac{(T_{S1} - T_{S2})}{r \left(\frac{1}{r_1} - \frac{1}{r_2}\right)}$$

Heat Transfer across the wall $q_r = -k(4\pi r^2) \frac{dT}{dr}$

$$\text{or, } q_r = +k(4\pi r^2) \frac{(T_{S1} - T_{S2})}{r^2 \left(\frac{1}{r_1} - \frac{1}{r_2}\right)} = \frac{4\pi k(T_{S1} - T_{S2})}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}$$

$$R_{t, \text{cond}} = \left(\frac{1}{r_1} - \frac{1}{r_2}\right) / 4\pi k = \frac{1}{4\pi k} \left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Critical Thickness of Insulation



The possible existence of an optimum insulation thickness for radial systems suggested by the competing effects associated with an increase in this thickness. In particular, although the conduction resistance increases with the addition of insulation, the convective resistance decreases due to increase in outer surface area.

$$q = \frac{T_i - T_\infty}{\frac{1}{2\pi L k} \ln\left(\frac{r_0}{r_i}\right) + \frac{1}{2\pi r_0 L} \cdot \frac{1}{h}}$$

$$= \frac{2\pi L (T_i - T_\infty)}{\frac{\ln(r_0/r_i)}{k} + \frac{1}{r_0 h}}$$

Now let us try with this expression to determine the outer radius of insulation r_0 which will minimize the heat transfer. The minimization condition is

$$\left. \frac{dq}{dr_0} \right| = 0 = \frac{-2\pi L (T_i - T_\infty) \left[\frac{1}{kr_0} - \frac{1}{hr_0^2} \right]}{\left[\frac{\ln(r_0/r_i)}{k} + \frac{1}{r_0 h} \right]^2}$$

Which gives

$$r_0 = \frac{k}{h}$$

This introduces the critical- radius-of-insulation concept. Up to this thickness of the outer-cylinder, the heat transfer will be increased by adding insulation. For outer radius greater than critical value an increase in insulation thickness will cause a decrease in heat transfer.

Conduction with Thermal Energy Generation (Radial Systems)

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) + \frac{q^*}{k} = 0 \quad (1)$$

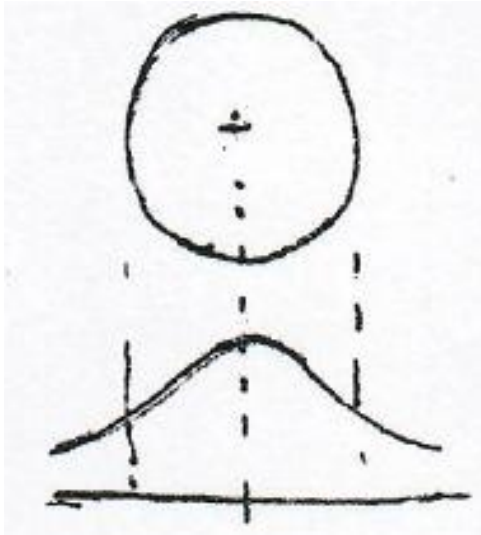
$$r \frac{dT}{dr} = -\frac{q^*}{2k} r^2 + c_1 \quad (2)$$

$$T(r) = -\frac{q^*}{4k} r^2 + c_1 \ln r + c_2 \quad (3)$$

- q^* $\equiv$ Uniformly distributed heat source
- Geometry is cylindrical
- Angular symmetry holds

Temperature distribution has to be determined.

The boundary condition are



$$\left. \frac{dT}{dr} \right|_{r=0} = 0 \quad \text{and} \quad T(r_0) = T_s$$

The first condition results from the symmetry of the situation.

From the first boundary condition and equation (2), we get $c_1 = 0$

Using the second boundary condition at $r = r_0$, we get

$$C_2 = T_s + \frac{q^*}{4k} r_0^2$$

$$T(r) = \frac{q^*}{4k} r_0^2 \left[1 - \left(\frac{r}{r_0} \right)^2 \right] + T_s$$

Now we can write

$$T(r) - T_s = \frac{q^*}{4k} r_0^2 \left(1 - \frac{r^2}{r_0^2} \right) \quad (4)$$

$$\begin{aligned} \frac{1}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) + \frac{q^*}{k} &= 0 \\ r \frac{dT}{dr} &= -\frac{q^*}{2k} r^2 + c_1 \\ T(r) &= -\frac{q^*}{4k} r^2 + c_1 \ln r + c_2 \end{aligned}$$

at the centre line

$$T_0 = \frac{q^* r_0^2}{4k} + T_s$$

$$\text{or } (T_0 - T_s) = \frac{q^* r_0^2}{4k} \quad (5)$$

From (4) and (5), the non-dimensional temp = $\frac{T(r) - T_s}{T_0 - T_s} = 1 - \left(\frac{r}{r_0}\right)^2$

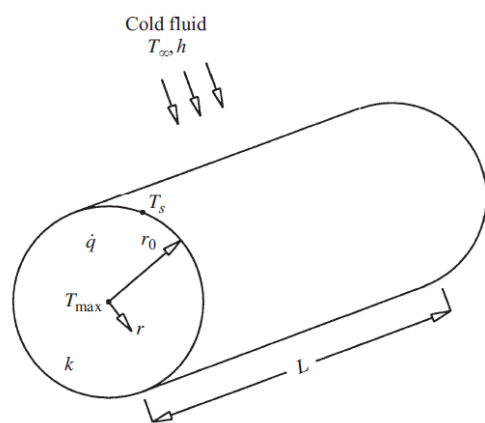
However, T_s in the above expression is unknown. It can be related to ambient temperature T_∞ from a overall energy balance.

$$q^* (\pi r_0^2 L) = h (2\pi r_0 L) (T_s - T_\infty)$$

Or

$$T_s = T_\infty + \frac{q^* r_0}{2h}$$

The foregoing approach may also be deployed to obtain the temperature distribution in solid spheres and cylindrical and spherical shells using the appropriate boundary conditions.



Solid Cylinder

Solid Sphere For the solid sphere,

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dT}{dr} \right) + \frac{\dot{q}}{k} = 0$$

$$\left. \frac{dT}{dr} \right|_{r=0} = 0 \quad \text{and} \quad T(r = r_0) = T_s$$

The solution is

$$T = T_s + \frac{\dot{q}}{6k} (r_0^2 - r^2)$$

The maximum temperature that occurs at the center of the sphere

$$T_{\max} = T_s + \frac{\dot{q} r_0^2}{6k}$$

and if the cooling at the outside surface is to a fluid at T_∞ via a heat transfer coefficient of h , the overall energy balance gives

$$T_s - T_\infty = \frac{\dot{q}r_0}{3h}$$

We will stop here today